PROJECTED WRITTEN NOTES FROM THE M 325K LECTURE ON TUESDAY, APRIL 25, 2024, ON SECTION 7.4 - MORE ON NOTES ON CARDINALITY.

CLASS # 27

The lecture began with a detailed presentation of the Solution -

Sec 7.4 #11:

$S = \{x \in \mathbb{R} \mid 0 < x < 1\}$

$V = \{x \in \mathbb{R} \mid 2 < x < 5\}$

To Prove: S and V have the same cardinality.

Proof: Define function $f: S \to V$ as follows:

FOR ALL $x \in S$, $f(x) = 3x + 2$.

This is well-defined, because if $x \in S$, then $0 < x < 1$ and so $0 < 3x < 3$, and so $2 < (3x+2) < 5$, and so $(3x+2) \in V$.

[f is one-to-one:] Suppose $u, v \in S$ are such that $f(u) = f(v)$. $f(u) = 3u+2$, $f(v) = 3v+2$.
Then $3u + 2 = 3v + 2$, $\therefore 3u = 3v$.
$\therefore u = v$. $\therefore f$ is one-to-one.

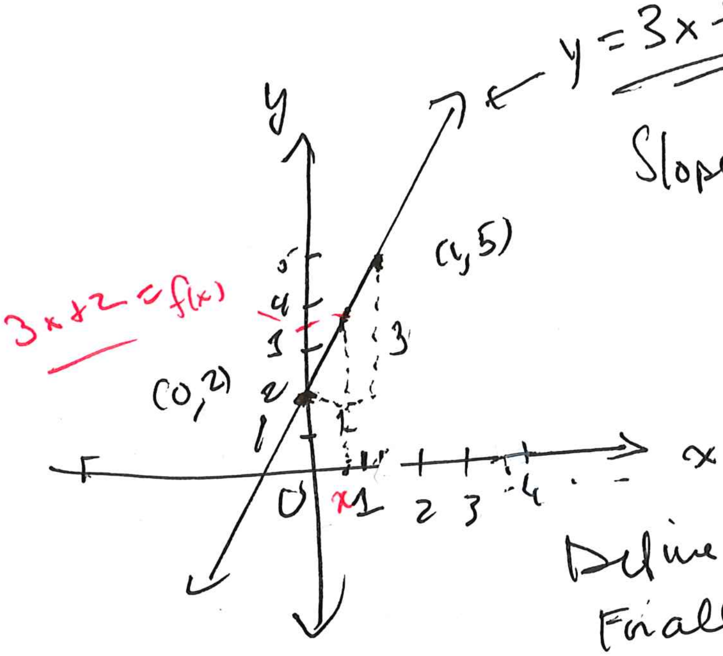
[f is onto:] Suppose y is any element in V.
Then $2 < y < 5$. Let $x = \frac{1}{3}(y-2)$.
[I must show that $x \in S$.]
Since $2 < y < 5$, $0 < y - 2 < 3$.
$\therefore 0 < \frac{1}{3}(y-2) < 1$. $\therefore 0 < x < 1$. $\therefore x \in S$.
Now, $f(x) = 3x + 2 = 3\left(\frac{1}{3}(y-2)\right) + 2$
$\therefore f(x) = (y-2) + 2 = y$. $\therefore f$ is onto.
$\therefore f$ is a one-to-one correspondence.
$\therefore$ S and V have the same cardinality. QED

DISCOVERING THE FUNCTION $f(x) = 3x + 2$.

With $S = (0,1)$

and $V = (2,5)$, How did come up with

$y = f(x) = 3x + 2$?



Slope $= \frac{3}{1} = 3 = m$

$y = mx + b$

$b = 2, m = 3$

Define $f: S \to V$ as follows:

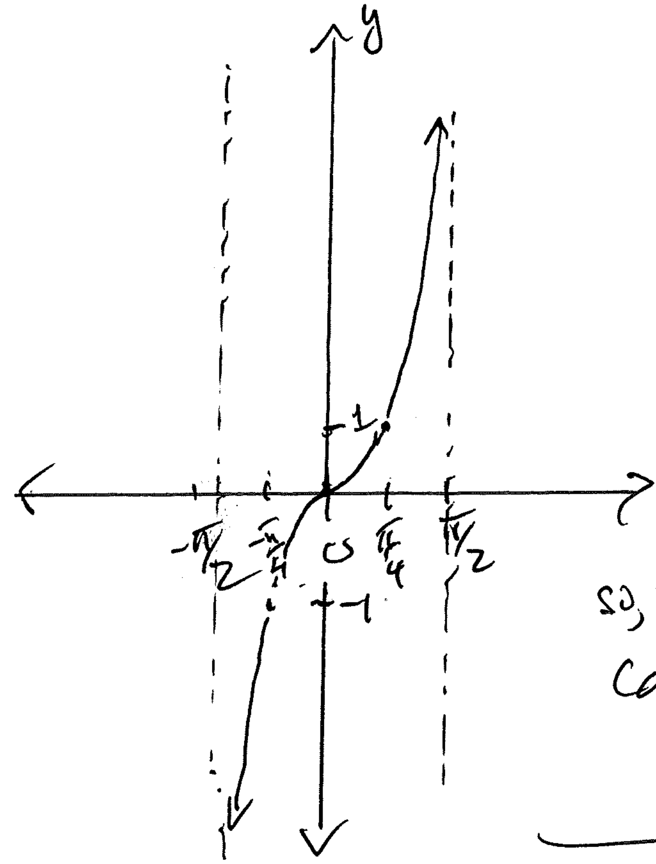
For all x, $0 < x < 1$,

$f(x) = 3x + 2$

Similarly we can show that $(0,1)$ has the same cardinality as any open interval (a,b)

In particular, $(0,1)$ and $(-\frac{\pi}{2}, \frac{\pi}{2})$ have same cardinality.

In particular, we can now show that $(-\pi/2, \pi/2)$ and $\mathbb{R} = (-\infty, \infty)$ have the same cardinality as follows:



For every real # x, $-\pi/2 < x < \pi/2$, define $f(x) = \tan(x)$.

Note that f is one-to-one and onto.

so, f is a one-to-one correspondence.

$\mathbb{R} = (-\infty, \infty)$ has the same cardinality as $(-\pi/2, \pi/2)$

$\therefore$ $\mathbb{R} = (-\infty, \infty)$ and $(0,1)$ have the same cardinality.

Important Theorems from "Notes on Cardinality"

Theorem 7.4.3: Any subset of a countable set is itself a countable set.

Ex: $\mathbb{Z}^+$ is countable (In fact, it is infinitely ctbl.)

The set $\mathbb{Z}^{EVEN-pos}$ of positive even integers is a subset of $\mathbb{Z}^+$, $\mathbb{Z}^{Even-pos} \subseteq \mathbb{Z}^+$.

By Thm 7.4.3, $\mathbb{Z}^{EVEN-pos}$ is countable, too.

Corollary 7.4.4: If a set A has a subset B, $B \subseteq A$, such that B is uncountable, then A is also uncountable.

Theorem (NIB) 12: The union of two countable sets is also a countable set.

Ex: We know that $\mathbb{Q}^+$ is countable (Thm (NIB) 13)

The mapping $r \to -r$ from $\mathbb{Q}^+$ to $\mathbb{Q}^-$ is a one-to-one correspondence. So, $\mathbb{Q}^-$ is is also countable. So $(\mathbb{Q}^+ \cup \mathbb{Q}^-)$ is countable by Theorem (NIB) 12.

$\{0\}$ is a finite set, so $\{0\}$ is a countable set. $\mathbb{Q} = (\mathbb{Q}^+ \cup \mathbb{Q}^-) \cup \{0\}$.

$\therefore$ By Theorem (NIB) 12, $\mathbb{Q}$ is countable.

The set of all Rational Numbers is Countable

A fact about real number

Every real number r such that $0 < r < 1$ can be represented by an infinite decimal expansion.

$r = 0.d_1 d_2 d_3 d_4 d_5 \ldots$, where, for each $i = 1, 2, 3, 4, \ldots$, $d_i \in \mathbb{Z}$ and $0 \le d_i \le 9$, such that

$$r = \sum_{i=1}^{\infty} d_i \left(\tfrac{1}{10}\right)^i,$$ a series that converges to r.

Ex: $\frac{1}{10}\pi = 0.314159\ldots$

$\frac{1}{3} = 0.3333\ldots$

$\frac{1}{2} = 0.50000\ldots$

$\frac{1}{2} = 0.499999\ldots$

If we omit from consideration those expansions that eventually $9999\ldots$ and we keep those expansions ending in $0000\ldots$, then every real # r, $0 < r < 1$, has a unique dicmal expansion.

This uniqueness means that

if $r_1 = 0.924333\ldots$

and $r_2 = 0.927333\ldots$

Then $r_1 \neq r_2$ because their decimal expansions differ in at least one decimal place.

ORIG

Proving that (0, 1) is an Uncountable Set

Theorem (NIB) 14:

Any infinite sequence of real numbers, all of which are between 0 and 1, will fail to include at least one real number between 0 and 1. That is:

For any infinite sequence $b_1, b_2, b_3, \ldots$ such that, $0 < b_i < 1$ for every $\mathbf{i} \in \mathbb{Z}^+$,

there exists some real number z, $0 < z < 1$, such that $b_i \neq z$ for all $\mathbf{i} \in \mathbb{Z}^+$.

Proof:

Suppose $b_1, b_2, b_3, \ldots$ is any infinite sequence of real numbers such that, for all $\mathbf{i} \in \mathbb{Z}^+$, $0 < b_i < 1$.

Line up the b_i's in a column with their decimal expansions, and ensure that in the decimal expansions, those expansions that end with 99999...
are replaced by equivalent expansions that end with 00000...

(each $\mathbf{d}_{ij}$ is the $\mathbf{j}^{th}$ digit in the expansion of b_i as found in the $\mathbf{i}^{th}$ row):

$$\begin{array}{lllllllll}
b_1 = 0 . & d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} & d_{17} & \ldots \\
b_2 = 0 . & d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} & d_{27} & \ldots \\
b_3 = 0 . & d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} & d_{37} & \ldots \\
b_4 = 0 . & d_{41} & d_{42} & d_{43} & d_{44} & d_{45} & d_{46} & d_{47} & \ldots \\
b_5 = 0 . & d_{51} & d_{52} & d_{53} & d_{54} & d_{55} & d_{56} & d_{57} & \ldots \\
\vdots & & & \vdots & & & & &
\end{array}$$

A real number $\mathbf{z}$ will be constructed so that $0 < z < 1$ and such that $\mathbf{z}$ does not appear in the sequence $b_1, b_2, b_3, \ldots$.

To define this number $\mathbf{z}$, we will focus on the $\mathbf{i}^{\text{th}}$ digit, d_{ii}, of the $\mathbf{i}^{\text{th}}$ term b_i for each $\mathbf{i} \in \mathbb{Z}^+$:

$$
\begin{array}{lllllllll}
b_1 & = 0 \,. & \boxed{d_{11}} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} & d_{17} \ \ldots \\
b_2 & = 0 \,. & d_{21} & \boxed{d_{22}} & d_{23} & d_{24} & d_{25} & d_{26} & d_{27} \ \ldots \\
b_3 & = 0 \,. & d_{31} & d_{32} & \boxed{d_{33}} & d_{34} & d_{35} & d_{36} & d_{37} \ \ldots \\
b_4 & = 0 \,. & d_{41} & d_{42} & d_{43} & \boxed{d_{44}} & d_{45} & d_{46} & d_{47} \ \ldots \\
b_5 & = 0 \,. & d_{51} & d_{52} & d_{53} & d_{54} & \boxed{d_{55}} & d_{56} & d_{57} \ \ldots \\
\ \vdots & & & & & \vdots & & &
\end{array}
$$

Then, we define this real number $\mathbf{z}$ as follows:

$$\mathbf{z} = 0.\, a_1\, a_2\, a_3\, a_4\, a_5\, a_6 \ \ldots, \text{ where, for each } \mathbf{i} \in \mathbb{Z}^+,$$

$$a_i = \begin{cases} 5 & \text{if } d_{ii} \neq 5 \\ 7 & \text{if } d_{ii} = 5 \end{cases}$$

By this process, the number $z = 0.\, a_1\, a_2\, a_3\, a_4\, a_5\, a_6 \ \ldots$ is uniquely defined.

The digits in the decimal expansion of z consist of 5's and 7's and the choice of each digit as 5 or 7 depends on the digits in the decimal expansions of the particular numbers in the sequence $b_1, b_2, b_3, \ldots$.

For example, suppose the sequence $b_1, b_2, b_3, \ldots$ begins as follows:

$$
\begin{array}{lcl}
b_1 & = & 0\,.\ \boxed{3}\ \ 8\ \ 2\ \ 6\ \ 7\ \ 5\ \ 2\ \ldots \\
b_2 & = & 0\,.\ 4\ \ \boxed{6}\ \ 1\ \ 9\ \ 5\ \ 8\ \ 5\ \ldots \\
b_3 & = & 0\,.\ 9\ \ 3\ \ \boxed{5}\ \ 8\ \ 6\ \ 1\ \ 2\ \ldots \\
b_4 & = & 0\,.\ 2\ \ 5\ \ 0\ \ \boxed{9}\ \ 4\ \ 3\ \ 7\ \ldots \\
b_5 & = & 0\,.\ 5\ \ 8\ \ 7\ \ 2\ \ \boxed{5}\ \ 9\ \ 0\ \ldots \\
\vdots & & \qquad\qquad\ \ \vdots
\end{array}
$$

Thus, the decimal expansion for z will begin as shown:

$$\boxed{z \;=\; \mathbf{0.55757\ldots}}$$

Now, it can be seen that $\mathbf{z \neq b_1}$
because their expansions differ in the 1st digit: $d_{11} = 3$, whereas $a_1 = 5$.

It can be seen that $\mathbf{z \neq b_2}$
because their expansions differ in the 2nd digit: $d_{22} = 6$, whereas $a_2 = 5$.

It can be seen that $\mathbf{z \neq b_3}$
because their expansions differ in the 3rd digit: $d_{33} = 5$, whereas $a_3 = 7$.

In the same way, **for all** $\mathbf{i \in \mathbb{Z}^+}$**,** $\mathbf{z \neq b_i}$ because their expansions differ in the i^{th} digit,

$a_i \neq d_{ii}$.

If the sequence $b_1, b_2, b_3, \ldots$ is different, the number z will be different,
but it will still be true that every digit in the decimal expansion of z is 5 or 7
(which guarantees that $0 < z < 1$)
and it will still be true that z differs from every number in the sequence.

Thus, in the general case, for the arbitrarily chosen sequence $b_1, b_2, b_3, \ldots,$
there exists a real number z such that $0 < z < 1$ and $b_i \neq z$, for all $i \in \mathbb{Z}^+$.

Q E D

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Theorem (NIB) 15: The interval $(0, 1) = \{ x \in \mathbb{R} \mid 0 < x < 1 \}$ is uncountable.

Proof: (proof-by-contradiction)

The interval $(0, 1)$ is either finite or infinite.

Certainly $(0, 1)$ is not finite because it contains the countably infinite set $\{ 1/2, 1/3, 1/4, 1/5, \ldots \}$ as a subset.

$\therefore$ The interval $(0, 1)$ is either a countably infinite set or an uncountable set.

Suppose that $(0, 1)$ is not uncountable, by way of contradiction.

Then, $(0, 1)$ is countably infinite.

$\therefore$ There exists a one-to-one correspondence $f : \mathbb{Z}^+ \rightarrow (0, 1)$.

[Note: f is one-to-one and onto; in particular, f is onto.]

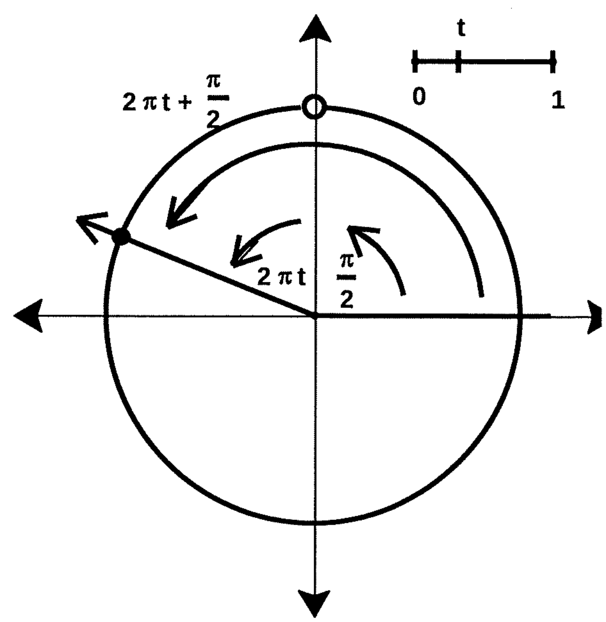
Define the infinite sequence $b_1, b_2, b_3, \ldots$ as follows:

For all $i \in \mathbb{Z}^+$, $b_i = f(i)$. Then, for all $i \in \mathbb{Z}^+$, $0 < b_i < 1$.

By Theorem (NIB) 14, there is some real number z, $0 < z < 1$, such that z does not appear in the sequence $b_1, b_2, b_3, \ldots$;
that is, for all $i \in \mathbb{Z}^+, b_i \neq z$. But that means that for all $i \in \mathbb{Z}^+$, $f(i) \neq z$.

Thus, f is not onto, which contradicts the assumption that f is a one-to-one correspondence. $\therefore$ The interval $(0, 1)$ is uncountable. Q E D

Theorem (NIB) 16: For Y equal to the set of all points on the Unit Circle minus the North Pole (0,1), the set Y is uncountable.



Proof: There are one-to-one correspondences between the Unit Circle minus the North Pole and the Interval $(0, 1)$ of real numbers. One such one-to-one correspondence is $f : (0, 1) \rightarrow Y$, defined as follows: For all $t \in (0, 1)$,

$$f(t) = \left(\cos\left(2\pi t + \frac{\pi}{2}\right), \sin\left(2\pi t + \frac{\pi}{2}\right) \right).$$

Therefore, interval $(0, 1)$ and Y have the same cardinality.

Since $(0, 1)$ is uncountable by Theorem (NIB) 15, Y is uncountable. Q E D

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When you are asked, on a homework problem or on a test, to prove that a particular set is uncountable, do not think that you need to use an argument with infinite decimal expansions of the numbers in a sequence. No, you should use a proof-by contradiction argument, as one shown here.

<u>Problem</u>: Let I = the set of all irrational numbers t such that $0 < t < 1$, that is, $t \in (0,1)$ and is irrational.

The set $\mathbb{Q} \cap (0,1)$ is the set of all rational numbers in $(0,1)$.

So, $(\mathbb{Q} \cap (0,1)) \cup I$ is the whole interval $(0,1)$, since I consists of the irrational numbers in $(0,1)$ and $\mathbb{Q} \cap (0,1)$ consists of all the rational numbers in $(0,1)$. Thus, $(\mathbb{Q} \cap (0,1)) \cup I = (0,1)$.

Here is the task: Prove that I is an uncountable set.

To Prove: The set I is uncountable.

Proof: Note that $(\mathbb{Q} \cap (0,1)) \subseteq \mathbb{Q}^+$ and it was proved earlier (Theorem (NIB)13) that $\mathbb{Q}^+$ has the same cardinality as $\mathbb{Z}^+$, which means that $\mathbb{Q}^+$ is a countable set. By Theorem 7.4.3, (which says that a subset of a countable set is itself a countable set), $\mathbb{Q} \cap (0,1)$ is countable. Recall that $(0,1)$ is uncountable. [We NTS that I is uncountable]

Suppose, by way of contradiction, that I is <u>NOT UNCOUNTABLE</u>. That means that <u>I is a countable set</u>, by assumption. Now, $(0,1) = (\mathbb{Q} \cap (0,1)) \cup I$, which is the union of two countable sets. By Theorem (NIB)12, $(\mathbb{Q} \cap (0,1)) \cup I$ is a countable set.

So, since $(0,1) = (\mathbb{Q} \cap (0,1)) \cup I$
and both $(\mathbb{Q} \cap (0,1))$ and I are countable
set, we conclude that $(0,1)$ is a countable
set by Theorem (N(B))12.
Thus, $(0,1)$ is a countable set AND
$(0,1)$ is an uncountable set,
which is a <u>CONTRADICTION</u>.

[Recall that this argument leading to a contradiction
began with the Supposition
"Suppose that I is NOT UNCOUNTABLE."]

Therefore, the set I of all the irrational numbers in the
interval $(0,1)$ <u>is</u> <u>uncountable</u>, by proof-by-contradiction.

Thus, we have proved that I is an uncountable set,
which was to have been proved. QED.

Earlier in the lecture, it was shown that
$(0,1)$ and the set of real numbers, $\mathbb{R}$, have the
same cardinality.
We later showed that $(0,1)$ is an uncountable set.
So, since $(0,1)$ and $\mathbb{R}$ have the SAME CARDINALITY,
we can conclude that $\mathbb{R}$ is an uncountable set also.

Also, we can apply Corollary 7.4.4 to prove that $\mathbb{R}$ is
uncountable because $\mathbb{R}$ contains the uncountable set $(0,1)$
as a subset, so $\mathbb{R}$ is also uncountable
by Corollary 7.4.4.